Kinetic theory of photon acceleration: Time-dependent spectral evolution of ultrashort laser pulses

L. Oliveira e Silva[1,2] and J. T. Mendonça[1]
[1]*GoLP, Centro de Física dos Plasmas, Instituto Superior Técnico, 1096 Lisboa Codex, Portugal*
[2]*Department of Physics and Astronomy, University of California, Los Angeles, California 90095*
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We investigate the evolution of the space- and time-dependent spectrum of an ultrashort laser pulse in the presence of relativistic plasma waves. A kinetic description of the laser pulse is introduced, generalizing the classical concept of the number of photons. The propagation equation for the generalized photon density is derived. The spectral deformation induced by a relativistic plasma perturbation in the laser pulse is also calculated. We also propose a new diagnostic technique for the electron density gradient, based on the analysis of the induced chirp in ultrashort laser pulses. [S1063-651X(98)10403-8]

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I. INTRODUCTION

Advances in laser technology allow now for the generation of high intensity ultrashort laser pulses [1]. The propagation of such pulses in a plasma is a subject of considerable importance due to the rich variety of nonlinear phenomena and to the applications of some of the mechanisms to advanced particle accelerators [2], the fast ignitor fusion concept [3], and new sources of tunable radiation [4,5].

Among these mechanisms, photon acceleration [5,6], or the frequency upshift of electromagnetic radiation by large amplitude plasma waves, has received considerable interest due to its potential use as a diagnostic tool for relativistic coherent plasma perturbations (plasma waves and ionization fronts), and as a possible source of tunable radiation and supercontinuum generation [7–9].

The usual theoretical descriptions of photon acceleration are based on two opposite approaches: the ray tracing, or Hamiltonian, formulation [7], and the usual plane wave Fourier expansion [8–10]. The first formulation describes in a general fashion the space-time dynamics of a wave packet, the classical analog of the photon, but does not describe the global spectrum and shape deformation of an electromagnetic pulse. The plane wave formulation allows for the derivation of transmission and reflection coefficients but fails to describe the localized nature of an ultrashort laser pulse and its space-time dynamics. The two descriptions give complementary, and yet incomplete, views of the same mechanism and fail to describe in a systematic way the full dynamics of the ultrashort laser pulse in the presence of a coherent plasma perturbation moving with phase velocity close to the speed of light c.

In this paper, we present a new kinetic description of photon acceleration based on a Klimontovich kinetic equation for the photons. In this formalism the classical analog of the number of photons evolves in phase space according to the Hamilton equations of motion for the photons, derived from the ray tracing equations. The space–wave-vector domain is fully described in a fashion similar to that currently employed in the characterization of ultrashort laser pulses [11]. The laser pulse spectrum along the extent of the pulse can then be examined, and the chirp induced by the perturbation can be determined. It is shown that the space–wave-number energy density is considerably deformed, giving rise to pulses that are not transform limited ($\Delta x \Delta k \gg 1$). This induced chirp can give us information about the local plasma density along the pulse extent. The usual frequency (upshift) (downshift) of photon acceleration is also observed, with the corresponding spectral broadening and electromagnetic energy increase (decrease).

This paper is organized as follows. In Sec. II, we derive the space wave-number distribution for the photons, generalizing the usual procedure for the classical number of photons of an electromagnetic plane wave. The main properties of the number of photons are also presented. The number of photons representation of a chirped Gaussian laser pulse is calculated, giving a clear picture of the most important features of this formalism. Section III includes the derivation of the time evolution equation for the space wave-number energy density starting from the energy conservation principle for the fields and the particles. The limit in which this equation is equivalent to a flux conservation equation for the phase space density, or a Klimontovich equation for the photons, is established. In Sec. IV, this formalism is applied to the propagation of a weak ultrashort laser pulse in the presence of a relativistic plasma wave. Analytic expressions for the frequency upshift and for the induced chirp of the laser pulses are derived. It is shown that the induced chirp can be easily related to the electron density perturbation. A considerable deformation of the spectrum is observed; the chirp of the laser pulse can be quite significant even for propagation in tenuous plasmas. Finally, in Sec. V, we state the conclusions.

II. NUMBER OF PHOTONS: PHASE-SPACE DEFINITION

The concept of the number of photons has been used in plasma physics since the 1960s [12,13]. This definition is accurate for plane waves but it is not valid for laser pulses. Previous attempts to generalize the number of photons to more general electromagnetic fields have been concerned with the mathematical formalism behind the derivation [14]. Our approach here will be focused on the applicability and limits of validity of the number of photons to the description of the time-dependent spectra of laser pulses.

We first generalize the concept of the number of photons in order to describe the space- and time-dependent spectrum of an ultrashort laser pulse propagating in dielectric media. The rate of change of the energy density of the electromagnetic field $\mathcal{E}_{\text{field}}$ is given by

$$\frac{\partial \mathcal{E}_{\text{field}}}{\partial t} = \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t}. \tag{1}$$

With the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ written in complex form, Eq. (1) can be written as

$$\frac{\partial \mathcal{E}_{\text{field}}}{\partial t} = \frac{1}{4}\left(\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}^*}{\partial t} + \mu_0 \boldsymbol{H}\cdot\frac{\partial \boldsymbol{H}^*}{\partial t} + \text{c.c}\right). \tag{2}$$

As usual [15] we neglect the rapidly varying terms $\boldsymbol{E}\cdot\dot{\boldsymbol{D}}$, $\boldsymbol{E}^*\cdot\dot{\boldsymbol{D}}^*$, $\boldsymbol{H}\cdot\dot{\boldsymbol{B}}$, and $\boldsymbol{H}^*\cdot\dot{\boldsymbol{B}}^*$. Equation (2) is obtained by a cycle average of Eq. (1). We also consider $\mu=\mu_0$. We now concentrate our efforts on the electric component of the field energy. Following the procedure outlined by Landau [15], we consider the constitutive relation $\boldsymbol{D}(\omega,\boldsymbol{k}) = \epsilon_0\epsilon(\omega,\boldsymbol{k})\boldsymbol{E}(\omega,\boldsymbol{k})$, where $\epsilon(\omega,\boldsymbol{k})$ is the relative permittivity of the dielectric medium, and we assume that the spectral content of the field is not very broad, i.e., $\Delta\omega/\omega_0 \ll 1$, with ω_0 the central frequency of the laser pulse. Under these assumptions, the contribution of the electric field related terms to Eq. (2) can be written as

$$\frac{\partial \mathcal{E}_{\text{electric}}}{\partial t} = \frac{\epsilon_0}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3}\frac{d\boldsymbol{k}'}{(2\pi)^3}\left.\frac{\partial \omega\epsilon(\omega,\boldsymbol{k})}{\partial\omega}\right|_{\omega=\omega_0} \times \boldsymbol{E}(\boldsymbol{k}',t)\cdot\frac{\partial \boldsymbol{E}^*(\boldsymbol{k},t)}{\partial t}\exp[-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}] + \text{c.c.} \tag{3}$$

Integrating Eq. (3) in all space (in this paper, when no explicit indication is given, all integrals extend from $-\infty$ to $+\infty$), and then integrating in $\boldsymbol{k}'$, we obtain

$$\frac{\partial}{\partial t}\int d\boldsymbol{r}\mathcal{E}_{\text{electric}} = \frac{\epsilon_0}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3}\left.\frac{\partial \omega\epsilon(\omega,\boldsymbol{k})}{\partial\omega}\right|_{\omega_0} \times \boldsymbol{E}(\boldsymbol{k},t)\cdot\frac{\partial \boldsymbol{E}^*(\boldsymbol{k},t)}{\partial t} + \text{c.c.} \tag{4}$$

If the electric field $\boldsymbol{E}(\boldsymbol{k},t)$ is written as

$$\boldsymbol{E}(\boldsymbol{k},t) = \int d\boldsymbol{r}'\boldsymbol{E}(\boldsymbol{r}',t)\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}') \tag{5}$$

and the complex conjugate $\boldsymbol{E}^*(\boldsymbol{k},t)$ is expressed as

$$\boldsymbol{E}^*(\boldsymbol{k},t) = \int d\boldsymbol{r}''\boldsymbol{E}^*(\boldsymbol{r}'',t)\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}'') \tag{6}$$

then, in the new integration variables $\boldsymbol{s}=\boldsymbol{r}''-\boldsymbol{r}'$ and $\boldsymbol{r}=(\boldsymbol{r}'+\boldsymbol{r}'')/2$, Eq. (4) reduces to

$$\frac{\partial}{\partial t}\int d\boldsymbol{r}\mathcal{E}_{\text{electric}} = \frac{\epsilon_0}{4}\frac{\partial}{\partial t}\int d\boldsymbol{r}\frac{d\boldsymbol{k}}{(2\pi)^3}\left.\frac{\partial \omega\epsilon(\omega,\boldsymbol{k})}{\partial\omega}\right|_{\omega_0}\mathcal{F}(\boldsymbol{k},\boldsymbol{r},t), \tag{7}$$

where $\mathcal{F}(\boldsymbol{k},\boldsymbol{r},t)$ obeys

$$\mathcal{F}(\boldsymbol{k},\boldsymbol{r},t) = \int d\boldsymbol{s}\boldsymbol{E}(\boldsymbol{r}-\boldsymbol{s}/2,t)\cdot\boldsymbol{E}^*(\boldsymbol{r}+\boldsymbol{s}/2,t)\exp(i\boldsymbol{k}\cdot\boldsymbol{s}), \tag{8}$$

which means that the electric component of the energy density verifies

$$\mathcal{E}_{\text{electric}}(\boldsymbol{r},t) = \frac{\epsilon_0}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3}\left.\frac{\partial \omega\epsilon(\omega,\boldsymbol{k})}{\partial\omega}\right|_{\omega_0}\mathcal{F}(\boldsymbol{k},\boldsymbol{r},t). \tag{9}$$

As for the magnetic component of the electromagnetic field energy, a similar procedure can also be used. From Eq. (2) it is obvious that the density of magnetic energy is simply $\mathcal{E}_{\text{magnetic}} = \mu_0/4\boldsymbol{H}\cdot\boldsymbol{H}^*$. Once again, if we express $\boldsymbol{H}$ as

$$\boldsymbol{H}(\boldsymbol{r},t) = \int \frac{d\boldsymbol{k}}{(2\pi)^3}d\omega\boldsymbol{H}(\boldsymbol{k},\omega)\exp(i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t) \tag{10}$$

and $\boldsymbol{H}^*$ as

$$\boldsymbol{H}^*(\boldsymbol{r},t) = \int \frac{d\boldsymbol{k}'}{(2\pi)^3}d\omega'\boldsymbol{H}^*(\boldsymbol{k}',\omega')\exp(i\boldsymbol{k}'\cdot\boldsymbol{r}-i\omega' t) \tag{11}$$

and using Maxwell equations ($\nabla\times\boldsymbol{E} = -\mu_0\partial\boldsymbol{H}/\partial t$), we obtain for the magnetic energy of the field

$$\mathcal{E}_{\text{magnetic}} = \frac{c^2\epsilon_0}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3}\frac{d\boldsymbol{k}'}{(2\pi)^3}d\omega d\omega' \times \frac{\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega)}{\omega}\cdot\frac{\boldsymbol{k}'\times\boldsymbol{E}^*(\boldsymbol{k}',\omega')}{\omega'} \times \exp\{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}-i(\omega-\omega')t\}. \tag{12}$$

Assuming, as before, that the spectrum is centered around ω_0 and that the spectral content is not very broad, we can integrate Eq. (12) in the variables ω and ω' and over all space to obtain

$$\int d\boldsymbol{r}\mathcal{E}_{\text{magnetic}} = \frac{\epsilon_0}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3}\frac{d\boldsymbol{k}'}{(2\pi)^3}\frac{c^2}{\omega_0^2} \times [\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},t)]\cdot[\boldsymbol{k}'\times\boldsymbol{E}^*(\boldsymbol{k}',t)](2\pi)^3 \times \delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{13}$$

Integrating over $\boldsymbol{k}'$, and using Eqs. (5) and (6), with the new variables $\boldsymbol{s}$ and $\boldsymbol{r}$ defined above, the expression for the magnetic component of the field energy reduces to

$$\mathcal{E}_{\text{magnetic}} = \frac{\epsilon_0}{4}\int \frac{d\boldsymbol{k}}{(2\pi)^3}\frac{\partial}{\partial\omega}\left.\left(\frac{-k^2c^2}{\omega} + \frac{(\boldsymbol{k}\cdot\boldsymbol{e}_E)^2}{\omega}\right)\right|_{\omega_0}\mathcal{F}(\boldsymbol{k},\boldsymbol{r},t), \tag{14}$$

where $\boldsymbol{e}_E=\boldsymbol{E}/|\boldsymbol{E}|$. It is now straightforward to define the energy density of the electromagnetic field as

$$\mathcal{E}_{\rm field}(\boldsymbol{r},t)=\frac{\epsilon_0}{4}\int\frac{d\boldsymbol{k}}{(2\pi)^3}\frac{\partial}{\partial\omega}\left(\omega\epsilon-\frac{k^2c^2}{\omega}+\frac{(\boldsymbol{k}\cdot\boldsymbol{e}_E)^2}{\omega}\right)\Bigg|_{\omega_0}\mathcal{F}(\boldsymbol{k},\boldsymbol{r},t). \quad (15)$$

This expression must be compared with the energy density written as a function of the number of photons $\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t)$:

$$\mathcal{E}_{\rm field}=\int\frac{d\boldsymbol{k}}{(2\pi)^3}2\hbar\omega(\boldsymbol{k},\boldsymbol{r},t)\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t). \quad (16)$$

Here $\hbar\omega$ is the energy of the individual photon, $d\boldsymbol{k}/(2\pi)^3$ is the number of possible photon states with momentum $\hbar\boldsymbol{k}$, $\omega(\boldsymbol{k},\boldsymbol{r},t)$ obeys the dispersion relation for the propagation medium, and the coefficient 2 accounts for the two polarizations. Comparing the quantumlike expression of Eq. (16) with Eq. (15), the natural definition of $\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t)$ is

$$\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t)=\frac{\epsilon_0}{8\hbar}\left(\frac{\partial D}{\partial\omega}\right)_{\omega_0}\mathcal{F}(\boldsymbol{k},\boldsymbol{r},t), \quad (17)$$

where $D\equiv 0$ is the dispersion relation for the dielectric medium. For a homogeneous plasma, in the absence of an external magnetic field, $D=1-c^2k^2/\omega^2-\omega_p^2/\omega^2$, with ω_p the electron plasma frequency. The number of photons can be regarded as a distribution function of quasiparticles, the photons, in phase space $(\boldsymbol{k},\boldsymbol{r})$. The physical meaning of both axes is clear: along the $\boldsymbol{k}$ axis , $\mathcal{N}$ represents the evolution of the field fast phase, or the $\boldsymbol{k}$ spectrum (fast time scale), while along the $\boldsymbol{r}$ axis we have the description of the slow amplitude of the electromagnetic field (slow time scale). Furthermore, $\boldsymbol{k}$ and $\boldsymbol{r}$ are now independent variables. The most important novelty of this formulation is the unified view of $\boldsymbol{k}$ and $\boldsymbol{r}$ space, which leads to a better understanding of the interplay between the spectral and spatial deformation of ultrashort laser pulses, and the connection between the two time scales. The number of photons written in this way is formally equivalent to the spectrum of the autocorrelation function, in the wave-vector–space domain. This function often occurs associated with the experimental characterization of ultrashort laser pulses with a time-dependent spectrum [11]. Some additional points must be made concerning the approximations involved in this derivation: (i) we have neglected rapidly varying terms in the energy density definition; (ii) we have assumed that the spectrum of the electromagnetic field is centered around a frequency ω_0 and the spectrum width $\Delta\omega$ is small compared with the frequency ω_0. For transform limited laser pulses, these approximations are valid if $\lambda_0/c\tau\ll 1$, where λ_0 is the central wavelength of the laser pulse and τ the pulse duration (for $\lambda_0=1$ μm, $\tau\gg 1$ fs).

For a plane wave, $\boldsymbol{E}=\boldsymbol{E_0}\exp(i\boldsymbol{k_0}\cdot\boldsymbol{r}-i\omega_0 t)$, the number of photons is simply

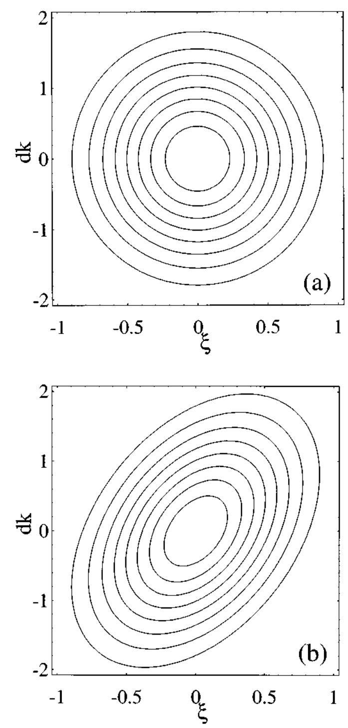

FIG. 1. Number-of-photons distribution for a Gaussian laser pulse propagating in vacuum in the phase space $\xi=(x-c\tau)/(c\tau)$, $dk=(k-k_0)c\tau$: (a) Without chirp, $b=0$; (b) with linear chirp $b=-0.45/\tau^2$ for the same amplitude contours.

$$\mathcal{N}_{\rm plane}(\boldsymbol{k})=\frac{\epsilon_0}{8\hbar}\frac{\partial D}{\partial\omega}|E_0|^2\delta(\boldsymbol{k}-\boldsymbol{k_0}), \quad (18)$$

which is just the usual definition of the number of photons found in the literature [12,13]. However, our more general approach allows for the definition of the number of photons even for short laser pulses. For a linearly chirped one-dimensional (1D) Gaussian laser pulse, propagating in a homogeneous medium along the x axis with group velocity v_g and phase velocity v_ϕ, duration τ, described by the electric field $\boldsymbol{E}=\boldsymbol{e}_E E_0\exp\{-(x-v_gt)^2/(c\tau)^2\}\exp\{-i[\omega_0(t-x/v_\phi)+b(t-x/v_\phi)^2]\}$, the number of photons verifies

$$\mathcal{N}_{\rm Gauss}(k,x,t)=\frac{\epsilon_0 c\tau\sqrt{\pi}|E_0|^2}{8\sqrt{2}\hbar}\frac{\partial D}{\partial\omega}\exp\left[-2\frac{(x-v_gt)^2}{(c\tau)^2}\right]\times\exp\left[-\frac{(c\tau)^2}{2}\left\{k-k_0+\frac{2b}{v_\phi}\left(\frac{x}{v_\phi}-t\right)\right\}^2\right]. \quad (19)$$

In Fig. 1, we plot the number of photons $\mathcal{N}$ given by Eq. (19) for a transform limited pulse $(b=0)$ and for a chirped pulse $b\neq 0$. The intuitive picture provided by the number of photons is clear: in linearly chirped pulses, the instantaneous spectral distribution, or the corresponding wave-number distribution derived from the dispersion relation $D\equiv 0$ for the photons, has a linear dependence with the relative position along the laser pulse extent, as shown by the spectral deformation in Fig. 1(b). The pulse width, spectral width, and chirp are calculated as the moments of the distribution function $\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t)$. In particular, the wave-vector chirp, i.e., how the local wave number changes along the laser pulse, at a given time t, can be expressed as

$$\langle \boldsymbol{k} \rangle_{r,t} = \frac{\int d\boldsymbol{k}\boldsymbol{k}\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t)}{\int d\boldsymbol{k}\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t)}. \tag{20}$$

The spatial intensity and the spectral intensity are obtained by calculating the following integrals:

$$\int d\boldsymbol{k}\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t) = \frac{\epsilon_0}{8\hbar}\left(\frac{\partial D}{\partial \omega}\right)_{\omega_0} |\boldsymbol{E}(\boldsymbol{r},t)|^2, \tag{21}$$

$$\int d\boldsymbol{r}\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t) = \frac{\epsilon_0}{8\hbar}\left(\frac{\partial D}{\partial \omega}\right)_{\omega_0} |\boldsymbol{E}(\boldsymbol{k},t)|^2. \tag{22}$$

The number of photons $\mathcal{N}$ defined above provides a full description of an ultrashort laser pulse, giving a particular emphasis to the internal evolution of the instant spectral and spatial distribution of the electromagnetic field.

III. FROM ENERGY CONSERVATION TO NUMBER-OF-PHOTONS CONSERVATION

In the previous section, the number of photons $\mathcal{N}$ was generalized in order to describe electromagnetic pulses. It is now important to describe how this distribution evolves in time. The most direct method to evaluate the time evolution of the number of photons would be to solve Maxwell equations for the electric field and, for each time t, to calculate the corresponding number-of-photons distribution. In this paper, we will not follow this approach since, apart from the new point of view, this would not add new insight to the problems of laser pulse propagation. Instead, starting from electromagnetic energy conservation for the field *and* for the particles of the medium, we will derive an equation for the evolution of the number of photons $\mathcal{N}$ in an unmagnetized plasma, which, for underdense plasmas, reduces to conservation of the number of photons. This approach leads to a generalization of the wave action conservation equation, which allows for the inclusion of all the nonlinear wave-wave and wave-particle interaction mechanisms in a natural way, starting from first principles.

The total energy in the electromagnetic field can be written as

$$W_{\text{field}} = \int d\boldsymbol{r} \frac{d\boldsymbol{k}}{(2\pi)^3} 2\hbar\omega(\boldsymbol{k},\boldsymbol{r},t)\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t), \tag{23}$$

where the integration in $\boldsymbol{r}$ is over all space. The energy transferred to the particles in the medium, which is already included in Eq. (23), is given by [16]

$$W_{\text{particles}} = \int d\boldsymbol{r} \frac{\boldsymbol{J}\cdot\boldsymbol{A}}{2}. \tag{24}$$

The critical point here is to determine the current density $\boldsymbol{J}$ due to the presence of the electromagnetic field, which is a function of the properties of the medium. We shall concentrate our efforts on an unmagnetized plasma. In this case, and for field intensities such that $a_0 = e|\boldsymbol{A}|/m_e c \ll 1$, the current density is simply written as

$$\boldsymbol{J} = -\epsilon_0 \omega_p^2(\boldsymbol{r},t)\boldsymbol{A}(\boldsymbol{r},t), \tag{25}$$

with $\omega_p = \sqrt{n_e e^2/\epsilon_0 m_e}$ the electron plasma frequency, and $n_e(\boldsymbol{r},t)$ the electron density of the plasma. Writing $\boldsymbol{A}$ in complex form, and neglecting rapidly varying terms, Eq. (24) now reduces to

$$W_{\text{particles}} = -\frac{\epsilon_0}{4}\int \mathrm{d}\boldsymbol{r}\omega_p^2(\boldsymbol{r},t)\boldsymbol{A}(\boldsymbol{r},t)\cdot\boldsymbol{A}^*(\boldsymbol{r},t). \tag{26}$$

We now assume that $\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}_0(\boldsymbol{r},t)\exp[iS(\boldsymbol{r},t)]$ such that $S(\boldsymbol{r},t)$ represents a fast phase and $\boldsymbol{A}_0$ is a slowly varying amplitude of $\boldsymbol{r}$ and t. Furthermore, we define the instantaneous frequency $\omega = -\partial S/\partial t$ and the instantaneous wave vector $\boldsymbol{k} = \nabla S$. Using these assumptions, in the Lorentz gauge, $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$, Eq. (26) verifies

$$W_{\text{particles}} = -\frac{\epsilon_0}{4}\int d\boldsymbol{r} \frac{\omega_p^2(\boldsymbol{r},t)}{\omega^2(\boldsymbol{k},\boldsymbol{r},t)}\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{E}^*(\boldsymbol{r},t). \tag{27}$$

For short laser pulses, using Parseval theorem and Eqs. (5) and (6), and following the procedure of Sec. II, the total energy transferred to the particles of the medium is written as

$$W_{\text{particles}} = -\int d\boldsymbol{r} \frac{d\boldsymbol{k}}{(2\pi)^3}\hbar\omega_p^2(\boldsymbol{r},t)\frac{1}{[\omega(\boldsymbol{k},\boldsymbol{r},t)]_{\boldsymbol{r}_0,\boldsymbol{k}_0}}\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t), \tag{28}$$

where $\boldsymbol{r}_0$ is the central position of the laser pulse, and the number of photons $\mathcal{N}$ is related with the electric field through Eq. (17). The dispersion relation for photons in an unmagnetized plasma was also employed, so that $(\partial D/\partial\omega) = 2/\omega$. It must be pointed out that we implicitly assumed that the density perturbation has a time scale and space scale such that mode coupling does not occur, i.e., $k_p \ll k_0$ and $\omega_p \ll \omega_0$ where $\lambda_p = 2\pi/k_p (\tau_p = 2\pi/\omega_p)$ is the typical spatial (temporal) scale of the perturbation. It must be stressed that the generalization of this procedure in order to include wave-wave interaction processes and, in particular, stimulated processes is straightforward. In fact, inserting the proper nonlinear current density in Eq. (24) will introduce nonlinear coupling between different regions of the phase space.

The total energy of the system obeys the equation

$$W_{\text{total}} = \int d\boldsymbol{r} \frac{d\boldsymbol{k}}{(2\pi)^3} 2\hbar\omega\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t). \tag{29}$$

In the absence of sources and/or sinks, the total energy is conserved. Hence,

$$\frac{d}{dt}[2\hbar\omega\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t)] = 0, \tag{30}$$

where d/dt is the convective derivative for the variables $\boldsymbol{k}$, $\boldsymbol{r}$, and t. From the linear dispersion relation $D \equiv 0$ and the ray tracing equation for ω $(d\omega/dt = \partial\omega/\partial t)$, the previous equation is expressed as

$$\frac{d\mathcal{N}}{dt} = -\frac{\mathcal{N}}{\omega}\frac{d\omega}{dt}, \tag{31}$$

For a stationary plasma, the right-hand side (rhs) of Eq. (31) vanishes, since $d\omega/dt=0$, and the number of photons $\mathcal{N}$ is conserved. In general, this will not occur, and the rhs contributes for the increase (decrease) of the number of photons if the frequency decreases, $d\omega/dt<0$, (increases, $d\omega/dt>0$). The rhs of Eq. (31) represents the first correction to the usual wave action conservation in inhomogeneous and nonstationary media. Previous derivations of the wave action conservation equation [17] have failed to identify the contribution of this correction term; the dissipative properties of the medium ($\mathrm{Im}D\neq 0$) contribute to the rhs of Eq. (31) in the form of a dissipative term. In our discussion, we have always considered $\mathrm{Im}D=0$.

For underdense plasmas, and assuming that $(1/\omega)d\omega/dt \sim 1/T$ (T is the time scale of the frequency variation), the number of photons $\mathcal{N}$ is conserved in the time scale $t_{\mathrm{cons}} \ll T$:

$$\frac{d\mathcal{N}}{dt}=\frac{\partial\mathcal{N}}{\partial t}+\frac{d\boldsymbol{k}}{dt}\cdot\frac{\partial\mathcal{N}}{\partial\boldsymbol{k}}+\frac{d\boldsymbol{r}}{dt}\cdot\frac{\partial\mathcal{N}}{\partial\boldsymbol{r}}\equiv O(1/T). \tag{32}$$

The photon dynamics is described by the ray tracing equations for $\boldsymbol{k}$ and $\boldsymbol{r}$, which are derived from the Hamiltonian $\omega\equiv\omega(\boldsymbol{k},\boldsymbol{r},t)$ [7], obtained by inverting the WKB dispersion relation $D(\omega,\boldsymbol{k},\boldsymbol{r},t)\equiv 0$. Equation (32) can then be written as

$$\frac{\partial\mathcal{N}}{\partial t}-\frac{\partial\omega}{\partial\boldsymbol{r}}\cdot\frac{\partial\mathcal{N}}{\partial\boldsymbol{k}}+\frac{\partial\omega}{\partial\boldsymbol{k}}\cdot\frac{\partial\mathcal{N}}{\partial\boldsymbol{r}}\equiv O(1/T)=0. \tag{33}$$

Equation (33) expresses the number-of-photons conservation in the phase space $(\boldsymbol{k},\boldsymbol{r})$ and it is equivalent to a Klimontovich kinetic equation for the distribution function $\mathcal{N}(\boldsymbol{k},\boldsymbol{r},t)$ because this microscopic distribution function represents the density of particles evolving in a six-dimensional space. Equation (33) describes the interplay between $\boldsymbol{k}$ space and $\boldsymbol{r}$ space through the time evolution of $\mathcal{N}$. The second term on the left-hand side of Eq. (33) describes group velocity dispersion, and the third term is responsible for the wavenumber spreading and/or compression due to the density gradient, which is also the contribution responsible for photon acceleration. With this approach, the physical mechanisms behind laser pulse propagation are clearly decoupled and the usual methods for solving flux conservation type equations can be used, while retaining the most important features of the propagation.

IV. PHOTON ACCELERATION BY A LASER WAKE FIELD

Previous attempts to use the equation for the conservation of the number of photons, or the wave kinetic equation, have been essentially concerned with stimulated scattering and turbulent scattering of plane waves in a plasma [12,13]. In this section, we apply the formalism to the interaction of an ultrashort laser pulse with a 1D relativistic wake field, created by a much stronger laser pulse, moving in the x direction. The linear dispersion relation, the essential ingredient of our discussion, is $\omega=\sqrt{k^2c^2+\omega_{p0}^2F[k_{p0}(x-v_\phi t)]}$, with $k=k_x$, $k_{p0}=\omega_{p0}/v_\phi\simeq\omega_{p0}/c$, and $F[k_{p0}(x-v_\phi t)]$ describes the normalized electron density modulation associated with the wake field. The velocity of the plasma perturbation, v_ϕ, is the group velocity of the stronger laser pulse, which creates the density perturbation, thus $v_\phi\sim c$. Using this dispersion relation, and calculating the contribution of the dispersion term in Eq. (33), we obtain the typical dispersion time scale for propagation in a homogeneous plasma $t_{\mathrm{disp}}=c\tau k_0\omega_0^2/(\Delta_k\omega_{p0}^2)$, where τ is the pulse duration, with a central frequency (wave number) ω_0 (k_0), and spectral width Δ_k. We will follow the laser pulse for times much shorter than t_{disp}, such that dispersion can be neglected. Moreover, we will consider propagation in the underdense regime $\omega_p/\omega\ll 1$. With these assumptions, the dispersion relation can be linearized around the central wave number k_0 (for $\Delta_k/k_0\ll 1$) in order to obtain

$$\omega(k,x,t)=k_0c+\frac{\omega_p^2}{2k_0c}+c(k-k_0)\left(1-\frac{\omega_p^2}{2k_0^2c^2}\right), \tag{34}$$

where $\omega_p^2=\omega_{p0}^2F[k_{p0}(x-v_\phi t)]$, and $k,k_0>0$. The frequency $\omega(k,x,t)$ plays the role of the Hamiltonian, which generates the ray tracing equations for the canonical variables k and x [7]. Introducing the normalized coordinates $\overline{t}=\omega_{p0}t$, $\overline{x}=k_{p0}x$, $\overline{k}=k/k_{p0}$, and performing the change of variables $\eta=\overline{x}-\beta_\phi\overline{t}$ and $\tilde{\epsilon}=k/k_0-2$, where $\beta_\phi=v_\phi/c$, the Hamiltonian for the new coordinates is

$$\Omega(\tilde{\epsilon},\eta,t)=\tilde{\epsilon}(1-\beta_\phi-\tilde{\omega}_p{}^2), \tag{35}$$

with $\tilde{\omega}_p{}^2=F(\eta)/(2\overline{k}_0^2)$, the new canonical momentum $\tilde{\epsilon}$, and the new canonical position η. The copropagation (counterpropagation) regime occurs for $\beta_\phi>0$ (<0). When defining the new position η, we have assumed that $v_\phi=\mathrm{const}$. In spite of the drastic approximations involved in deriving Eqs. (34) and (35), the most important features are retained: the possibility of photon acceleration and the dependence of the group velocity on the local electron density. In the new variables, Eq. (33) is written as

$$\frac{\partial\mathcal{N}}{\partial\overline{t}}+(1-\beta_\phi-\tilde{\omega}_p{}^2)\frac{\partial\mathcal{N}}{\partial\eta}+\tilde{\epsilon}\frac{\partial\tilde{\omega}_p{}^2}{\partial\eta}\frac{\partial\mathcal{N}}{\partial\tilde{\epsilon}}=0, \tag{36}$$

which can be integrated explicitly by the method of characteristics for several dependencies $F(\eta)$. The general solution to Eq. (36) can be written implicitly as [18]

$$\mathcal{N}(\tilde{\epsilon},\eta,t)=\mathcal{N}_i\left(k_i=[\tilde{\epsilon}_0(\tilde{\epsilon},\eta,\overline{t})+2]k_0,\right.$$
$$\left.x_i=\eta_0(\tilde{\epsilon},\eta,\overline{t})\frac{c}{\omega_{p0}}\right), \tag{37}$$

where $\mathcal{N}_i(k_i,x_i)$ is the initial photon distribution function. The functions $\tilde{\epsilon}_0(\tilde{\epsilon},\eta,\overline{t})$ and $\eta_0(\tilde{\epsilon},\eta,\overline{t})$ are obtained by inverting the solutions of the ray tracing equations, corresponding to the Hamiltonian in Eq. (35):

$$\frac{d\eta}{d\overline{t}}=\frac{\partial\Omega}{\partial\tilde{\epsilon}}=1-\beta_\phi-\tilde{\omega}_p^2, \tag{38}$$

$$\frac{d\tilde{\epsilon}}{d\bar{t}}=-\frac{\partial\Omega}{\partial\eta}=\tilde{\epsilon}\frac{\partial\tilde{\omega}_p{}^2}{\partial\eta}, \tag{39}$$

$$\frac{d\Omega}{d\bar{t}}=\frac{\partial\Omega}{\partial\bar{t}}=0. \tag{40}$$

Since we are considering dispersionless propagation, the solution of Eq. (38) does not depend on $\tilde{\epsilon}$, and so $\eta_0(\eta,\bar{r})$ is also independent of $\tilde{\epsilon}$. Assuming an initial distribution $\mathcal{N}_i$ symmetric in respect to k_0, we can calculate the wave-number chirp using Eq. (20) and conservation of the new Hamiltonian $\Omega(\tilde{\epsilon},\eta)$:

$$\langle k\rangle_{\eta,\bar{t}}=k_0[2-\Theta(\eta,\bar{t})], \tag{41}$$

where $\Theta(\eta,\bar{t})$ obeys the equation

$$\Theta(\eta,\bar{t})=\frac{1-\beta_\phi-\tilde{\omega}_p{}^2(\eta_0)}{1-\beta_\phi-\tilde{\omega}_p{}^2(\eta)}. \tag{42}$$

Equation (41) can be expressed for $\tilde{\epsilon}$ as $\langle\tilde{\epsilon}\rangle_{\eta,\bar{t}}=-\Theta(\eta,\bar{t})$. The analysis of Eq. (41) shows that a pulse injected in a region with constant electron density, such that $\tilde{\omega}_p{}^2(\eta_0)=C_i=\text{const}$ for any η_0 along the pulse, will be chirped only in the region of electron density gradient; when it arrives at a new region where $\tilde{\omega}_p{}^2(\eta)=C_f=\text{const}$ for the entire laser pulse, no chirp will be observed. However, even if chirp is not present (a transform limited pulse remains transform limited after interaction with the density gradient), the number-of-photons distribution will be distorted. In fact, the spectral width of the laser pulse verifies:

$$\Delta_k^2=\langle k^2\rangle-\langle k\rangle^2=\frac{\int dkdxk^2\mathcal{N}(k,x,t)}{\int dkdx\mathcal{N}(k,x,t)}-\left[\frac{\int dkdxk\mathcal{N}(k,x,t)}{\int dkdx\mathcal{N}(k,x,t)}\right]^2. \tag{43}$$

It can be easily shown for transform limited Gaussian pulses that, after the interaction with the electron density gradient, the spectral width is

$$\Delta_k^2=\Delta_{ki}^2\Theta^2(\eta,\bar{t})=\Delta_{ki}^2\left(\frac{1-\beta_\phi-C_i}{1-\beta_\phi-C_f}\right)^2. \tag{44}$$

Furthermore, the central wave number $\langle k\rangle$ of the laser pulse is shifted by a factor

$$\delta k=\langle k\rangle-k_0=\frac{\int dkdxk\mathcal{N}(k,x,t)}{\int dkdx\mathcal{N}(k,x,t)}-k_0$$

$$=k_0\left(1-\frac{1-\beta_\phi-C_i}{1-\beta_\phi-C_f}\right). \tag{45}$$

This corresponds to the usual maximum frequency shift and to pulse compression predicted for the interaction of a laser pulse and an electron beam in the underdense regime.

We now consider the simplest model for the electron density perturbation describing a laser wake field: $F(\eta)=1+\delta\sin(\eta)$, where δ represents the amplitude of the electron density oscillation. In this case, the ray tracing equations (38)–(39) can be fully integrated, so that $\eta_0(\tilde{\epsilon},\eta,\bar{t})$ and $\tilde{\epsilon}_0(\tilde{\epsilon},\eta,\bar{t})$ obey the equations

$$\eta_0(\tilde{\epsilon},\eta,\bar{t})=\mathcal{G}^{-1}(\mathcal{G}(\eta)-\bar{t}+\bar{t}_0), \tag{46}$$

$$\tilde{\epsilon}_0(\tilde{\epsilon},\eta,\bar{t})=\frac{\tilde{\epsilon}}{\Theta(\eta,\eta_0)}, \tag{47}$$

where the function $\mathcal{G}(\eta)$ verifies

$$\mathcal{G}(\eta)=\frac{2}{\sqrt{p^2-q^2}}\arctan\left[\frac{q\cos(\eta/2)+p\sin(\eta/2)}{\cos(\eta/2)\sqrt{p^2-q^2}}\right], \tag{48}$$

where $p=1-\beta_\phi-1/(2\bar{k}_0{}^2)$ and $q=-\delta/(2\bar{k}_0{}^2)$. Inserting Eq. (46) and Eq. (48) in Eq. (36) we have the complete evolution of the number of photons in the presence of a laser wake field. In Fig. 2, we plot the number of photons for several propagation times, in a copropagation configuration $k>0,\beta_\phi>0$, where it is possible to see the deformation of the laser pulse spectrum while evolving in the wake field. This deformation is compared with the chirp described by Eq. (41). As predicted, the chirp follows very closely the electron density perturbation. This is evident from Eq. (41). For copropagation in a relativistic wake field $\beta_\phi\sim1$. Furthermore if $(1-\beta_\phi)\omega_0^2/\omega_p^2\ll1$, the chirp is given by

$$\langle\tilde{\epsilon}\rangle_{\eta,\bar{t}}\simeq-\frac{\tilde{\omega}_p{}^2(\eta_0)}{\tilde{\omega}_p{}^2(\eta)}=-\frac{1+\delta\sin(\eta_0)}{1+\delta\sin(\eta)}. \tag{49}$$

This means that even for underdense plasmas, the chirp induced by the wake field can be significant, and it is of the order of the electron density modulation δ. On the other hand, for counterpropagation $\beta_\phi\sim-1$, the chirp induced by the wake field is of the order $(\omega_p/\omega_0)^2$.

Another important feature is the possibility of relating the maximum chirp with the gradient of the electron density, giving rise to another diagnostic method for this fundamental parameter of laser plasma particle accelerators. Unlike previous techniques based on the photon acceleration concept [19,20], this new diagnostic method does not rely on the measurement of the phase-frequency shift induced by the plasma wave on the pulse centroid; it is based on the chirp of the probe beam, i.e., the different phase-frequency shifts that the pulse experiences along its extent. The maximum chirp induced by the laser wake field, i.e., the maximum of $d\langle k\rangle/d\eta$ calculated in the centroid of the pulse, occurs for the maximum of $d\Theta/d\eta$, which is equivalent to the condition

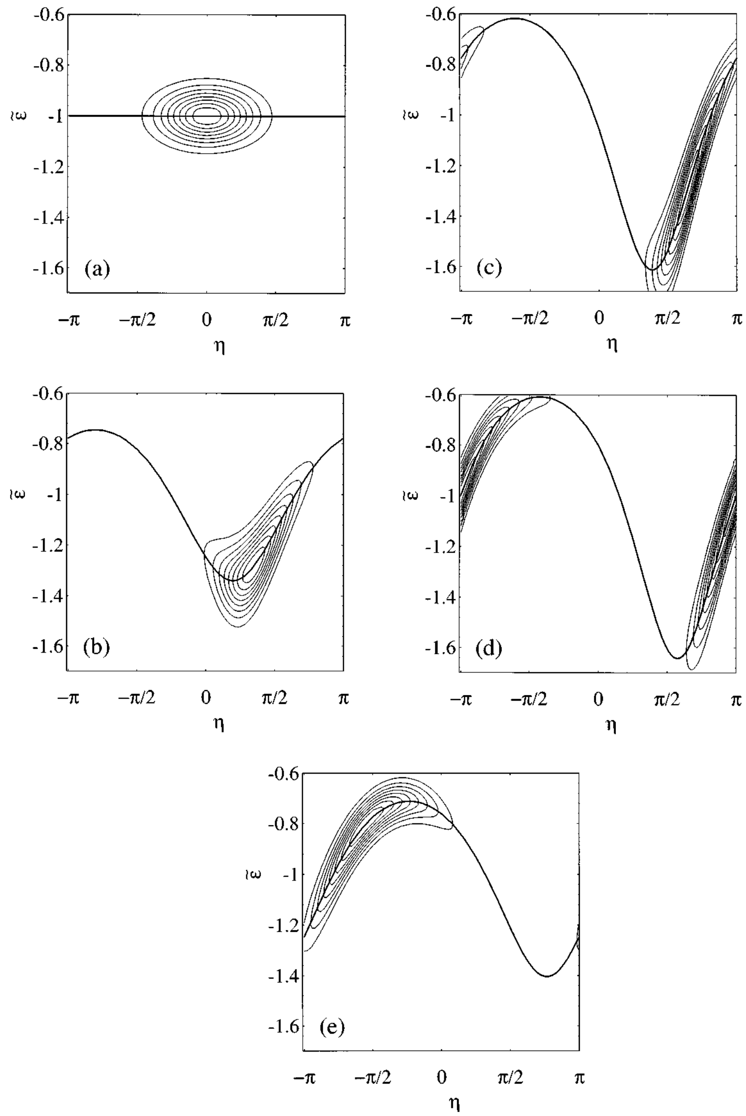

FIG. 2. Time evolution of the number-of-photons distribution for an initially transform limited Gaussian pulse in the phase space $(\tilde{\epsilon},\eta)$: (a) $\overline{t}=0$, (b) $\overline{t}=250$, (c) $\overline{t}=500$, (d) $\overline{t}=750$, (e) $\overline{t}=1000$ with $\tau\omega_{p0}=1$, $k_0/k_p=10$, $\delta=0.25$, and $\beta_\phi=0.99$, for the same amplitude contours. The chirp predicted by Eq. (41) is also plotted (solid line).

$$\max\left(\frac{d\Theta}{d\eta}\right)_{\eta_c}=\max\left(\frac{\Theta^2}{1-\beta_\phi-\tilde{\omega}_p{}^2(\eta_0)}\frac{d\tilde{\omega}_p{}^2(\eta)}{d\eta}\right)_{\eta_c}$$

$$\simeq C_1\max\left|\left(\frac{d\tilde{\omega}_p{}^2(\eta)}{d\eta}\right)_{\eta_c}\right|, \tag{50}$$

where only first order terms of δ have been retained, $\delta\ll 1$, and $C_1=1-\beta_\phi-\omega_p^2(\eta_0)=\text{const}$ is calculated for the pulse central position η_c. This means that by probing the laser wake field with weak ultrashort laser pulses, and analyzing the chirp of the probe pulses, it is possible to determine the maximum electron density gradient of the electron plasma wave. Other techniques [19] must rely on a continuous probing of the wake field structure to derive this fundamental parameter. The linear chirp for the wake field described by $F(\eta)$ is then related to the parameter b in Eq. (19) by $b\simeq \pm\beta_\phi\omega_{p0}\omega_0\delta/2$. For laser pulse durations $\tau\omega_{p0}\sim 1$, the parameter b can be easily measured by autocorrelation techniques, down to electron density modulations as low as $\delta=0.01$. Therefore, the determination of the maximum of the chirp is sufficient to determine the electron density modulation δ. Some indeterminacy associated with the velocity of the wake field β_ϕ is still present. However, as suggested by Dias *et al.* [20], a comparison between copropagation and counterpropagation can circumvent this indeterminacy due to the unknown interaction length of the probe pulse with the moving electron density perturbation.

The considerably large chirp induced by the wake field in the copropagation scheme will also give rise to a significant spectral and spatial deformation. This can be observed in Figs. 3 and 4. In these figures, we present the spatial inten-

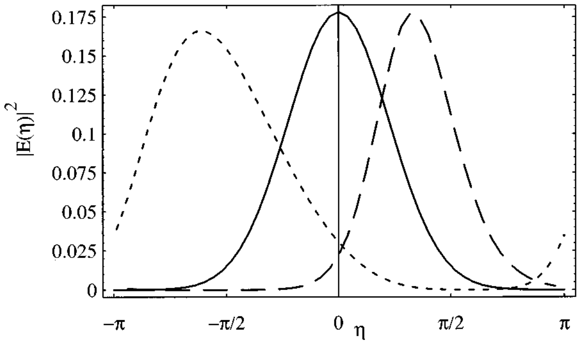

FIG. 3. Electric field spatial intensity (arbitrary units) as a function of η for the same conditions as in Fig. 2, calculated from Eq. (21): $\overline{t}=0$ (solid line), $\overline{t}=250$ (long-dashed line), $\overline{t}=1000$ (short-dashed line).

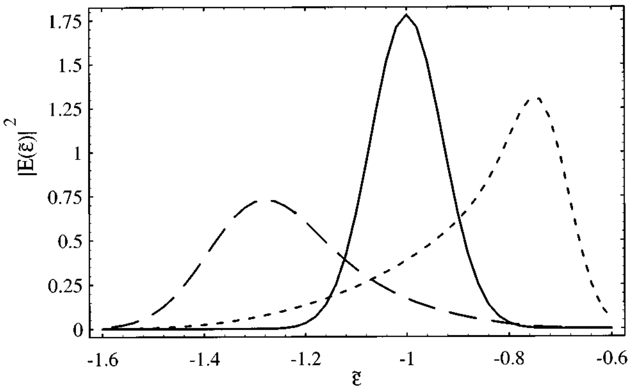

FIG. 4. Electric field spectral intensity (a.u.) as a function of $\widetilde{\epsilon}$ given by Eq. (22) for the same conditions as in Fig. 2: $\overline{t}=0$ (solid line), $\overline{t}=250$ (long-dashed line), $\overline{t}=1000$ (short-dashed line).

sity and the spectral intensity of the laser pulse for the same conditions of Fig. 2. The spatial intensity and the spectral intensity are calculated from the $\mathcal{N}$ distribution using Eqs. (21) and (22). The evolution of the spatial intensity shows the wake field regions where the laser pulse is ''accelerated,'' $\nabla F(\eta)<0$ [total pulse energy increases, $W_{\text{field}}(\overline{t}=1000)=1.14\times W_{\text{field}}(\overline{t}=0)$], and the regions where the laser pulse is ''decelerated'' $\nabla F(\eta)>0$ [total energy decreases, $W_{\text{field}}(\overline{t}=250)=0.765\times W_{\text{field}}(\overline{t}=0)$].

The energy increase (decrease) of the laser pulse is accompanied by a wave number up (down) shift (Fig. 4) so that the number of photons remains conserved. This energy increase (decrease) is accompanied by a pulse spreading (compression) in the regions of acceleration (deceleration), while the maximum electric field remains more or less constant, thus, increasing (decreasing) the total field energy. This feature could not be predicted if the less realistic plane wave number of photons distribution of Eq. (18) was used: in this case, number-of-photons conservation implies that a frequency increase necessarily leads to an amplitude increase of the electric field, which is not the case for electromagnetic pulses, as we have just mentioned.

The spectral intensity also evolves in a very peculiar way. The central wave number decreases (increases) when the laser pulse is decelerated (accelerated). A nonsymmetric wave number spreading is also observed (Fig. 4): this is due to the nonlinear chirp induced by the wake field (Fig. 2).

The source (sink) for the field energy is the energy stored by the plasma electrons. In our approach, the energy exchange between the electrons and the field does not affect the plasma oscillation. A self-consistent description of the wake field and the laser pulse dynamics would lead to a photon Landau damping scenario [21], which is only relevant when the ponderomotive force of the laser pulse in the plasma cannot be neglected (intense short laser pulses).

Finally, it must be pointed out that the dynamics of the laser pulse is recurrent; after a full libration in the wake field, the laser pulse will recover its initial characteristics. This is a consequence of the time-independent nature of the Hamiltonian $\Omega(\widetilde{\epsilon},\eta)$ [7].

V. CONCLUSIONS

We have derived an expression for the number of photons, and its time evolution equation, starting from first principles, namely, the energy conservation for the system field +plasma electrons. Our generalization allows for the extension of this concept to ultrashort laser pulses. The evolution equation can be reduced to a Klimontovich kinetic equation in phase space, in the underdense regime $(\omega_p/\omega_0 \ll 1)$. However, the full evolution equation contains a correction term of the order of $(\omega_p/\omega_0)^2$ reflecting the influence of the time-dependent plasma frequency, which, as far as we know, was never identified before. This result indicates that further exploration of the relation between energy conservation and conservation of the number of photons is necessary in order to clarify the limits of validity of the photon conservation equation.

Our new formalism was then applied to study a typical mechanism where a time-dependent spectrum is observed: photon acceleration. We have calculated the chirp induced by a wake field in copropagation and counterpropagation. In copropagation the induced chirp can be significant, and depends essentially on the electron density modulation associated with the wake field, while for counterpropagation the induced chirp is negligible $[O(\omega_p^2/\omega_0^2)]$. Based on the analysis of the induced chirp, a new diagnostic technique of the electron density gradient was proposed. The evolution of the spatial intensity and the spectral intensity was also analyzed, confirming the large induced chirp associated with the photon acceleration effect. The total electromagnetic energy variation associated with photon acceleration was also analyzed. It was shown that this increase (decrease) of the total energy leads to a pulse expansion (compression).

The results presented here can also be extended to the interaction of short pulses with electron beams or ionization fronts. In the latter case, a generalization of Eq. (31) is necessary since the current density is no longer given by Eq. (25). This will be the subject of a future publication.

The number-of-photons formalism introduced here puts a strong emphasis on the space- and time-dependent characteristics of the laser pulse, giving a clear picture of the interplay between the long and the short time scales. This approach can lead to a better theoretical understanding of the several nonlinear phenomena occurring in the interaction of ultrashort laser pulses with plasmas.

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